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A thermoelastic problem for interface cracks with contact zones

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Abstract

A problem of thermoelasticity for a set of cracks situated on the interface of two dissimilar isotropic solids under a combined tension–shear loading and uniform heat flow is considered. The cracks considered are assumed to be completely open, partially closed with frictionless thermally-conducted contact zones and completely closed. By means of the complex-function method the problem is reduced to a non-homogeneous Dirichlet–Riemann boundary value problem, which has been solved in closed form. For the determination of the contact zone lengths the condition of smooth closure of the crack faces has been used and a set of transcendental equations has been obtained. The closed-form expressions for the stresses on the interface and the derivatives of the displacement jumps across the interface as well as the stress intensity factors have been obtained. The numerical examples for a crack with one contact zone and for a crack with two contact zones have been presented. For these cases the dependencies of the stress intensity factors and the relative contact zone lengths with respect to the coefficient of the intensity of thermal and mechanical loading for various thermoelastic constants are presented, and a comparison of the results concerning the crack with one and two contact zones has been performed.

Keywords: Interface crack; Thermoelasticity; Stress intensity factor

1. Introduction

The problem of an interface crack attracted a considerable attention in the literature because of its importance for applications. Starting from a pioneer paper by Williams (1959) numerous essential results have been obtained by using classical (open crack) model possessing an oscillating singularity at the crack tips. Particularly in the frame of this assumption an interface crack with partially insulated crack surfaces in an isotropic bimaterial under heat flow has been considered analytically by Brown and Erdogan (1968). A heat transmission coefficient between the adjusted crack surfaces has been taken into account by Kuo (1990), and the stress intensity factors for an insulated and partially insulated interface crack in an isotropic

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bimaterial under heat flow has been determined by Lee and Shul (1991) and Lee and Park (1995), respectively.

To eliminate the interpenetration of crack faces near the crack tips a new model which takes into account the crack faces contact zones has been initiated by Comninou (1977). Concerning a pure mechanical loading this model has been analytically studied and further developed in the papers by Atkinson (1982), Simonov (1985), Gautesen and Dundurs (1987), Dundurs and Gautesen (1988) Gautesen (1992, 1993), Loboda (1993). As for a thermal loading a contact zone model for an interface crack in an anisotropic bimaterial has been analytically considered by Herrmann and Loboda (2001) and for a piezoelectric bimaterial it has been studied by Qin and Mai (1999). It is worth to note that Herrmann and Loboda (2001) considered a single crack with one contact zone and in the paper by Qin and Mai (1999) thermally insulated contact zones have been assumed and the problem has been reduced to the singular integral equation that has been solved numerically. The axisymmetric problem for a thermally insulated penny-shaped interface crack with a contact zone under tension-thermal loading has been studied by means of the method of singular integral equations by Martin-Moran et al. (1983) and Barber and Comninou (1983). However, to the author's knowledge a plane problem for an arbitrary number of interface cracks with contact zones in an isotropic bimaterial under thermomechanical loading has not been studied earlier.

In the present paper an exact analytical solution for a set of interface cracks which can be completely open, partially closed with frictionless thermally-conducted contact zones and completely closed under a thermomechanical loading is presented. The transcendental equations for the determination of the real contact zone lengths are formulated. The results of the numerical analysis have been presented for a single crack with two and one contact zones.

2. Formulation of the problem

Consider a plane problem for a bimaterial composed of two dissimilar isotropic semi-infinite spaces (in the case of plane strain) or planes (in the case of plane stress) with thermomechanical parameters E_k (Young's moduli), ν_k (Poisson's ratios), k_k (coefficients of thermal conductivity) and α_k (coefficients of thermal expansion), where the subscript $k = 1, 2$ means that the respective term refers to the upper and lower half-planes, respectively. A set of cracks is assumed on the interface. Under the influence of a uniform tension–shear loading (σ–τ) and uniform heat flux (q_2^∞) applied at infinity, the cracks may partially or completely open. The open parts of the cracks will be regarded as thermally insulated and the contact regions as frictionless and perfectly thermally conducted. The points of transition from the bonded interface to contact regions are denoted as a_i $(i = 1, 2, \ldots, I)$, from the separation to the contact regions, b_j $(j = 1, 2, \ldots, J)$ and from the bond to separation regions, c_n $(n = 1, 2, \ldots, N)$ (Fig. 1). The stresses $\sigma_{11}^{(1)\infty}$ and $\sigma_{11}^{(2)\infty}$ shown in Fig. 1 are applied at infinity in order for the continuity condition at infinity will be satisfied. They should satisfy the following equality (Rice and Sih, 1965)

$$\frac{1+\kappa_1}{\mu_1}\sigma_{11}^{(1)\infty} - \frac{1+\kappa_2}{\mu_2}\sigma_{11}^{(2)\infty} = \left[\frac{\kappa_2 - 3}{\mu_2} - \frac{\kappa_1 - 3}{\mu_1}\right]\sigma,$$

in which

$$\mu_k = \frac{E_k}{2(1+\nu_k)}, \quad \kappa_k = \begin{cases} 3 - 4\nu_k & \text{for a plane strain,} \\ \frac{3-\nu_k}{1+\nu_k} & \text{for a plane stress.} \end{cases}$$

As it can be seen in the following, these stresses do not influence on the thermomechanical fields along the interface and, consecutively, on the SIFs and contact zone lengths.

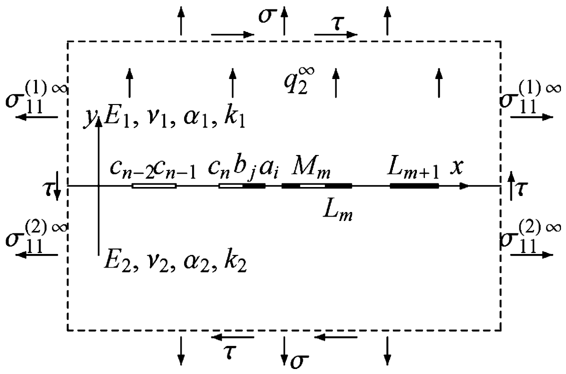

Fig. 1. Geometry of the problem.

The continuity conditions at $y = 0$ in the Cartesian coordinates x, y can be written as

$$\begin{cases} [\sigma_{22}(x)] - \mathrm{i}[\sigma_{12}(x)] = 0, \\ [q_2(x)] = 0, \end{cases} \quad x \in L + M + U \begin{cases} [u_1(x)] + \mathrm{i}[u_2(x)] = 0, & x \in U, \\ [T(x)] = 0, & x \in U + L, \end{cases} \tag{1a, b}$$

where $\sigma_{12}^{(k)}(x,y)$, $\sigma_{22}^{(k)}(x,y)$, $u_j^{(k)}(x,y)$, $q_j^{(k)}(x,y)$, $T^{(k)}(x,y)$ are the components of the shear and normal stresses, the displacements, the heat flux and the temperature, respectively; the superscripts $k = 1$ and $k = 2$ denote that associated quantity refers to the upper and lower half-planes, respectively. The brackets denote here the jump of the correspondent function across the interface, i.e. $[f(x)] = f^{(1)}(x,0) - f^{(2)}(x,0)$. The interface regions M, L are defined as follows:

$$M = \bigcup_{n=1}^{(N+J)/2} M_n, \quad L = \bigcup_{n=1}^{(I+J)/2} L_n;$$

U, M_n and L_n denote the bond, gaps (open parts of the cracks) and contact zones, respectively.

The conditions (1a) mean that the fields of stresses and heat flux $(\sigma_{12}, \sigma_{22}, q_2)$ have no discontinuity on the entire interface $(L + M + U)$. Next conditions (1b) mean that the fields of displacements and temperature are continuous on the bonded parts of the interface (U) and the temperature is continuous in the contact regions (L) as well since we assume that the contact regions are perfectly thermally conducted.

The boundary conditions in the contact regions (L) and on the open parts of the crack surfaces (M) can be presented in the form

$$\begin{cases} \sigma_{12}^{(1)}(x,0) = 0, \\ [u_2(x)] = 0, \end{cases} \quad x \in L \quad \begin{cases} \sigma_{22}^{(1)}(x,0) - \mathrm{i}\sigma_{12}^{(1)}(x,0) = 0, \\ q_2^{(1)}(x,0) = 0, \end{cases} \quad x \in M. \tag{1c, d}$$

The conditions (1c) express that contact interaction is frictionless and the crack faces is in contact for $x \in L$. Expressions (1d) mean that the open parts of the crack faces are unloaded and thermally insulated.

It is expedient to represent the stresses $\sigma_{ij}^{(k)}(x,y)$ and heat flux $q_i^{(k)}(x,y)$ fields in the form

$$\sigma_{ij}^{(k)}(x,y) = \sigma_{ij}^{(k)\infty} + \sigma_{ij}^{*(k)}(x,y), \quad q_i^{(k)}(x,y) = q_i^{(k)\infty} + q_i^{*(k)}(x,y), \tag{1e}$$

where $\sigma_{22}^{(k)\infty} = \sigma$, $\sigma_{12}^{(k)\infty} = \tau$, $q_2^{(k)\infty} = q_2^\infty$, $q_1^{(k)\infty} = 0$, $\sigma_{ij}^{*(k)}(x,y)$ and $q_i^{*(k)}(x,y)$ are the fields perturbed by cracks which vanish at infinity. Allowing for (1e), the boundary conditions (1c, d) for $\sigma_{ij}^{*(k)}(x,y)$ and $q_2^{*(k)}(x,y)$, acquire the following form

$$\begin{cases} \sigma_{12}^{(1)}(x,0) = -\tau, \\ [u_2(x)] = 0, \end{cases} \quad x \in L \quad \begin{cases} \sigma_{22}^{(1)}(x,0) - \mathrm{i}\sigma_{12}^{(1)}(x,0) = -\sigma + \mathrm{i}\tau, \\ q_2^{(1)}(x,0) = -q_2^\infty, \end{cases} \quad x \in M. \tag{1f, g}$$

Taking into account that in the following the perturbed fields only will be considered the asterix (*) here and in the following in the designation of thermomechanical fields is dropped out.

It should be noted, that the lengths of contact zones (position of the points b_j) are unknown initially and are dependent on the applied load and crack interaction. In the following analysis these positions will be defined from the additional condition which are equivalent to those suggested by Comninou (1977).

3. Complex-function representation of thermomechanical fields

For a plane problem of thermoelasticity, the following Muskhelishvili's (1975) complex-function representation for stress and displacement fields is used

$$\begin{cases} \sigma_{22}^{(k)}(x,y) - \mathrm{i}\sigma_{12}^{(k)}(x,y) = \Phi_k(z) + \overline{\Phi_k(z)} + z\overline{\Phi_k'(z)} + \overline{\Psi_k(z)}, \\ 2\mu_k[u_1^{(k)}(x,y) + \mathrm{i}u_2^{(k)}(x,y)] = \kappa_k\varphi_k(z) - z\overline{\Phi_k(z)} - \overline{\psi_k(z)} + (1+\kappa_k)H_k \int \theta_k(z)\,\mathrm{d}z, \end{cases} \tag{2}$$

where $\mu_k = E_k/2(1+\nu_k)$;

$$H_k = \begin{cases} \frac{\alpha_k E_k}{2(1-\nu_k)} & \text{for a plane strain,} \\ \frac{\alpha_k E_k}{2} & \text{for a plane stress,} \end{cases} \qquad \kappa_k = \begin{cases} 3-4\nu_k & \text{for a plane strain,} \\ \frac{3-\nu_k}{1+\nu_k} & \text{for a plane stress,} \end{cases}$$

$\Phi_k(z) = \varphi_k'(z)$, $\Psi_k(z) = \psi_k'(z)$, $\theta_k(z)$ are the analytical functions of the complex variable $z = x + \mathrm{i}y$ in the upper ($k = 1$) and lower ($k = 2$) half-planes. The overbar denotes complex conjugation. The prime (′) stands for differentiation on the corresponded argument.

The temperature and the heat flux components which define by the formulae

$$q_1^{(k)}(x,y) = -k_k \frac{\partial T^{(k)}(x,y)}{\partial x}, \quad q_2^{(k)}(x,y) = -k_k \frac{\partial T^{(k)}(x,y)}{\partial y}$$

can be expressed in terms of the functions $\theta_k(z)$ as

$$T^{(k)}(x,y) = \theta_k(z) + \overline{\theta_k(z)}, \quad q_1^{(k)}(x,y) - \mathrm{i}q_2^{(k)}(x,y) = -2k_k\theta_k'(z). \tag{3a, b}$$

For the further analysis it is expedient to introduce the following functions

$$\omega_k(z) = z\bar{\varphi}_k'(z) + \bar{\psi}_k(z), \tag{4a}$$

analytical in the respective half-planes. After replacing z by $\bar{z}$ in (4a), one has

$$\overline{\psi_k(z)} = \omega_k(\bar{z}) - \bar{z}\overline{\varphi_k'(z)}, \tag{4b}$$

that gives after differentiation

$$\overline{\psi_k'(z)} = \omega_k'(\bar{z}) - \overline{\varphi_k'(z)} - \bar{z}\overline{\varphi_k''(z)}. \tag{4c}$$

By inserting (4b) and (4c) into (2) and denoting $\omega_k'(z) = \Omega_k(z)$, Eq. (2) can be rewritten in the form

$$\begin{cases} \sigma_{22}^{(k)}(x,y) - \mathrm{i}\sigma_{12}^{(k)}(x,y) = \Phi_k(z) + (z-\bar{z})\overline{\Phi_k'(z)} + \Omega_k(\bar{z}), \\ 2\mu_k[u_1^{(k)}(x,y) + \mathrm{i}u_2^{(k)}(x,y)] = \kappa_k\varphi_k(z) + (\bar{z}-z)\overline{\Phi_k(z)} - \omega_k(\bar{z}) + (1+\kappa_k)H_k \int \theta_k(z)\,\mathrm{d}z. \end{cases} \tag{5}$$

Using expressions (3a), (3b) and (5), satisfying the continuity conditions (1a) and (1b) and denoting the boundary values of the functions analytical in the upper and lower half-planes by the superscripts "+" and "−" for $y \to +0$ and $y \to -0$, respectively, one can obtain

$$\begin{cases} \Phi_1^+ - \Omega_2^+ = \Phi_2^- - \Omega_1^-, & x \in L + M + U, \\ k_1\theta_1'^+ + k_2\overline{\theta_2'}^+ = k_2\theta_2'^- + k_1\overline{\theta_1'}^-, & x \in L + M + U \end{cases} \tag{6a}$$

and

$$\begin{cases} \frac{\kappa_1}{2\mu_1}\Phi_1^+ + \frac{1}{2\mu_2}\Omega_2^+ + \frac{(1+\kappa_1)H_1}{2\mu_1}\theta_1^+ = \frac{\kappa_2}{2\mu_2}\Phi_2^- + \frac{1}{2\mu_1}\Omega_1^- + \frac{(1+\kappa_2)H_2}{2\mu_2}\theta_2^-, & x \in U, \\ \theta_1'^+ - \overline{\theta_2'}^+ = \theta_2'^- - \overline{\theta_1'}^-, & x \in U + L. \end{cases} \tag{6b}$$

The argument x has been dropped in the last equations for the sake of shortness. Since two sides of (6a) and (6b) represent the boundary values of two analytical functions in the respective half-planes, therefore, they can be analytically extended into the entire plane, so that one can write

$$A(z) = \begin{cases} k_1\theta_1'(z) + k_2\overline{\theta}'_2(z), & y > 0, \\ k_2\theta_2'(z) + k_1\overline{\theta}'_1(z), & y < 0, \end{cases} \tag{7a}$$

$$B(z) = \begin{cases} \Phi_1(z) - \Omega_2(z), & y > 0, \\ \Phi_2(z) - \Omega_1(z), & y < 0. \end{cases} \tag{7b}$$

The function $A(z)$ and $B(z)$ are analytical in the whole plane, and

$$G(z) = \begin{cases} \frac{\kappa_1}{2\mu_1}\Phi_1(z) + \frac{1}{2\mu_2}\Omega_2(z) + \frac{(1+\kappa_1)H_1}{2\mu_1}\theta_1(z), & y > 0, \\ \frac{\kappa_2}{2\mu_2}\Phi_2(z) + \frac{1}{2\mu_1}\Omega_1(z) + \frac{(1+\kappa_2)H_2}{2\mu_2}\theta_2(z), & y < 0, \end{cases} \tag{8a}$$

$$\theta'(z) = \begin{cases} \theta_1'(z) - \overline{\theta}'_2(z), & y > 0, \\ \theta_2'(z) - \overline{\theta}'_1(z), & y < 0, \end{cases} \tag{8b}$$

where $G(z)$ and $\theta'(z)$ are analytical in the whole plane except the segments $L + M$ and M, respectively. Taking into account that the functions are vanished at infinity, according to Liouvill's theorem one can derive $A(z) = B(z) = 0$, and the expressions (7a)–(8b) can be rewritten as follows

$$\theta_k(z) = k_m\theta(z)/(k_1 + k_2), \quad \bar{\theta}_k(z) = -k_m\theta(z)/(k_1 + k_2), \ \ m = \{1 \text{ if } k = 2 \text{ and } 2 \text{ if } k = 1\} \tag{9}$$

and

$$\begin{cases} \Phi_1(z) = \Omega_2(z) = g\Big[G(z) - \frac{(1+\kappa_1)H_1}{2\mu_1}\theta_1(z)\Big], & y > 0, \\ \Phi_2(z) = \Omega_1(z) = g\gamma\Big[G(z) - \frac{(1+\kappa_2)H_2}{2\mu_2}\theta_2(z)\Big], & y < 0, \end{cases} \tag{10}$$

where $g = 2\mu_1\mu_2/(\kappa_1\mu_2 + \mu_1)$, $\gamma = (\kappa_1\mu_2 + \mu_1)/(\kappa_2\mu_1 + \mu_2)$.

Inserting (9) and (10) into (3b) and (5) and writing the obtained expressions for the interface ($y = 0$) result in

$$\begin{cases} \sigma_{22}^{(1)}(x,0) - \mathrm{i}\sigma_{12}^{(1)}(x,0) = g\Big[G^+(x) + \gamma G^-(x) - \tilde{g}[\theta^+(x) + \tilde{\gamma}\theta^-(x)]\Big], \\ [u_1'(x)] + \mathrm{i}[u_2'(x)] = G^+(x) - G^-(x), \\ q_2^{(1)}(x,0) = -\mathrm{i}\tilde{k}[\theta'^+(x) + \theta'^-(x)], \end{cases} \tag{11}$$

where $\tilde{k} = k_1k_2/(k_1 + k_2)$, $\tilde{g} = 2\delta_1\tilde{k}$, $\tilde{\gamma} = \gamma\delta_2/\delta_1$ and δ_k are the distortivities (Martin-Moran et al., 1983) of the respective half-planes which define by the formulae

$$\delta_k = \begin{cases} \frac{\alpha_k(1+\nu_k)}{k_k} & \text{for a plane strain,} \\ \alpha_k/k_k & \text{for a plane stress.} \end{cases}$$

4. Formulation and solution of the boundary value problem

By use of the formula (11_3) from the condition ($1d_2$) the following Hilbert problem can be derived

$$\theta'^{+}(x) + \theta'^{-}(x) = -\mathrm{i}q_2^{\infty}/\tilde{k}, \quad x \in M. \tag{12}$$

According to Muskhelishvili (1975) the solution to this problem can be presented as

$$\theta'(z) = -\frac{\mathrm{i}q_2^{\infty}}{2\tilde{k}f(z)}\left[\frac{1}{\pi\mathrm{i}}\int_M \frac{f^{+}(x)\,\mathrm{d}x}{x-z} + R(z)\right], \tag{13}$$

where

$$f(z) = \prod_{j=1}^{J}(z-b_j)^{1/2}\prod_{n=1}^{N}(z-c_n)^{1/2}, \quad R(z) = \sum_{n=0}^{p} R_n z^n, \;\; p = (J+N)/2,$$

R_n are constants to be determined from the condition at infinity and single-valuedness conditions which can be written as

$$\theta'(z) = \mathrm{O}(z^{-1}), \;\; z \to \infty, \quad \int_{M_n} [\theta'^{+}(x) - \theta'^{-}(x)]\,\mathrm{d}x = 0, \;\; n = 1, \ldots, p.$$

Making use of these conditions one can derive the following expressions for R_n

$$R_p = 1, \quad R_{p-1} = \frac{1}{2}\left(\sum_{j=1}^{J} b_j + \sum_{n=1}^{N} c_n\right), \quad \mathbf{r} = \mathbf{W}^{-1}\mathbf{v}, \tag{14}$$

in which

$$\mathbf{r} = [R_0, \ldots, R_{p-2}]^T, \quad W_{nj} = \int_{M_n} \frac{x^j\,\mathrm{d}x}{f^{+}(x)}, \quad v_n = -\int_{M_n} \frac{x^{p-1}(x + R_{p-1})}{f^{+}(x)}\,\mathrm{d}x, \;\; n = 1, \ldots, p-2.$$

Having evaluated the integral in (13) the expression for $\theta'(z)$ acquires the form

$$\theta'(z) = -\frac{\mathrm{i}q_2^{\infty}}{2\tilde{k}}\left(1 - \frac{f^{\infty}(z) - R(z)}{f(z)}\right), \tag{15a}$$

where $f^{\infty}(z)$ is a principal part of the expansion of the function $f(z)$ at infinity. Integrating Eq. (15a) gives

$$\theta(z) = -\frac{\mathrm{i}q_2^{\infty}}{2\tilde{k}}[z - f_t(z)], \quad f_t(z) = \int \frac{f^{\infty}(z) - R(z)}{f(z)}\,\mathrm{d}z. \tag{15b}$$

It is worth to note that $f_t^{+}(x) = -f_t^{-}(x)$, $x \in M$. The function $f_t(z)$ has the following behaviour at infinity

$$f_t(z) = z + \beta_0 + \beta_1/z + \mathrm{O}(z^{-2}), \quad z \to \infty. \tag{16}$$

Satisfying the remaining boundary conditions (1f) and (1g) by means of (11), taking into account (15b) and (16) and introducing the new function

$$F(z) = G(z) - \mathrm{i}q_2^{\infty}\delta_1[\gamma_* f_t(z) - \gamma^*(z + \beta_0)] + \tilde{\sigma} - \mathrm{i}\tilde{\tau}, \tag{17a}$$

in which $\gamma_* = (1-\tilde{\gamma})/(1-\gamma)$, $\tilde{\sigma} = \sigma/g(1+\gamma)$ and $\tilde{\tau} = \tau/g(1+\gamma)$ lead to

$$\begin{cases} F^+(x) + \gamma F^+(x) = 0, & x \in M, \\ \operatorname{Im} F^{\pm}(x) = q^*(x), & x \in L, \end{cases} \tag{17b}$$

where $q^*(x) = q_2^\infty \delta_1 \gamma_0 f_t(x)$, $\gamma_0 = (\gamma^* - \gamma_*)$, $\gamma^* = (1+\tilde{\gamma})/(1-\gamma)$.

The derived problem (17b) is a combined non-homogeneous Dirichlet–Riemann boundary value problem. A general solution of such a problem was given by Nakhmein and Nuller (1988), and concerning the present case a solution unbounded at all points a_i, b_j, c_n can be written as

$$F(z) = X(z)F_*(z), \quad F_*(z) = P(z) + I_1(z) + \mathrm{i}Y(z)[Q(z) + I_2(z)], \tag{18a}$$

in which

$$X(z) = \frac{\mathrm{e}^{\mathrm{i}\varphi(z)}}{f(z)p(z)}, \quad p(z) = \prod_{l=2}^{(I+J)/2} (z-d_l), \;\; f(z) = \prod_{j=1}^{J} (z-b_j)^{1/2} \prod_{k=1}^{N} (z-c_k)^{1/2}, \tag{18b}$$

$$\varphi(z) = -\varepsilon Z(z) \int_M \frac{\mathrm{d}x}{Z(x)(x-z)} - \mathrm{i}Z(z) \sum_{n=1}^{(I+J)/2} \int_{L_n} \frac{h_n(x)}{Z^+(x)(x-z)}\,\mathrm{d}x, \quad \varepsilon = \frac{\ln\gamma}{2\pi}, \tag{18c}$$

$$Z(z) = \prod_{i=1}^{I} (z-a_i)^{1/2} \prod_{j=1}^{J} (z-b_j)^{1/2}, \quad Y(z) = \prod_{i=1}^{I} (z-a_i)^{-1/2} \prod_{j=1}^{J} (z-b_j)^{1/2}, \tag{18d}$$

$$I_1(z) = \frac{1}{\pi} \int_L \frac{q_1(x)}{x-z}\,\mathrm{d}x, \quad q_1(x) = -\cos[\pi h_k(x)] q^*(x) f(x) p(x) \operatorname{sh}\tilde{\varphi}(x), \;\; x \in L_k, \tag{18e}$$

$$I_2(z) = \frac{1}{\pi\mathrm{i}} \int_L \frac{q_2(x)}{Y^+(x)(x-z)}\,\mathrm{d}x, \quad q_2(x) = \cos[\pi h_k(x)] q^*(x) f(x) p(x) \operatorname{ch}\tilde{\varphi}(x), \;\; x \in L_k, \tag{18f}$$

$$\tilde{\varphi}(x) = \pm\mathrm{i}[\varphi^{\pm}(x) - \pi h_n(x)], \quad x \in L_n, \tag{18g}$$

$$h_1(x) = n_1^*, \;\; x \in L_1, \quad h_l(x) = n_l^* + U(d_l - x), \;\; x \in L_l, \;\; l = 2, \ldots, (I+J)/2,$$

$U(x) = \{1 \text{ if } x > 0;\ 0 \text{ if } x < 0\}$ is the Heaviside's function. Besides, n_k^* are integers, $d_l \in L_l$ are unknown poles of $X(z)$ to be determined from the finiteness conditions for $\varphi(z)$ at infinity, which can be written as

$$\varepsilon \int_M \frac{x^{l-2}\,\mathrm{d}x}{Z(x)} + \mathrm{i} \sum_{n=1}^{(I+J)/2} \int_{L_n} \frac{h_n(x)}{Z^+(x)} x^{l-2}\,\mathrm{d}x = 0, \quad l = 2, \ldots, (I+J)/2. \tag{19}$$

Moreover the polynomials $P(z)$ and $Q(z)$ with real coefficients C_k, D_k have the following form

$$P(z) = \sum_{k=1}^{m+1} C_k z^k, \;\; m = J + (I+N)/2 - 1, \quad Q(z) = \sum_{k=1}^{n+1} D_k z^k, \;\; n = I + (N+J)/2 - 1, \tag{20a}$$

and the mentioned coefficients are to be determined from the finiteness conditions of $F(z)$ at poles d_l

$$F_*^-(d_l) = 0, \quad F_*'^-(d_l) = 0, \tag{20b}$$

condition at infinity, which, according to (16) and (17a), can be written as

$$F(z) = \tilde{\sigma} - \mathrm{i}\tilde{\tau} + p^* f_t(z) + \mathrm{O}(z^{-1}), \quad z \to \infty \tag{20c}$$

and single-valuedness conditions, which, according to the second formula in (11) and (17a), can be presented in the form

$$\int_{a_n'}^{a_n''}[F^+(x)-F^-(x)]\,\mathrm{d}x=-\mathrm{i}q_2^\infty\delta_1\gamma_*\int_{a_n'}^{a_n''}[f_t^+(x)-f_t^-(x)]\,\mathrm{d}x, \tag{20d}$$

here a_n' and a_n'' denote the left and right crack tips, respectively.

Taking the boundary values of (18a)–(18f) by means of Plemelj's formulae and using (17a) and (11) the following expressions for the stresses on the interface and for the derivative of the displacement jumps of the crack faces can be derived:

$$\frac{\sigma_{22}^{(1)}(x)}{g(1+\gamma)}=\frac{\cos[\pi h_k(x)]}{\mathrm{ch}(\pi\varepsilon)p(x)}\left[\frac{P(x)+\mathrm{p.v.}I_1(x)}{f_1(x)}\mathrm{ch}(\tilde{\varphi}(x)-\pi\varepsilon)+\mathrm{i}\frac{Q(x)+\mathrm{p.v.}I_2(x)}{f_2^+(x)}\mathrm{sh}(\tilde{\varphi}(x)-\pi\varepsilon)\right],\quad x\in L_k, \tag{21}$$

$$\frac{\sigma_{22}^{(1)}-\mathrm{i}\sigma_{12}^{(1)}}{g(1+\gamma)}=\frac{\exp[\mathrm{i}\varphi(x)]}{p(x)}\left[\frac{P(x)+I_1(x)}{f_1(x)}+\mathrm{i}\frac{Q(x)+I_2(x)}{f_2(x)}\right]-\mathrm{i}q_2^\infty\delta_1\gamma_0 f_t(x),\quad x\in U, \tag{22}$$

$$[u_2'(x)]=\frac{2\,\mathrm{ch}(\pi\varepsilon)}{p(x)}\left[\frac{P(x)+I_1(x)}{\mathrm{i}f_1^+(x)}\cos\varphi^*(x)-\frac{Q(x)+I_2(x)}{\mathrm{i}f_2^+(x)}\sin\varphi^*(x)\right],\quad x\in M, \tag{23}$$

$$[u_1'(x)]=2\frac{\cos[\pi h_k(x)]}{p(x)}\left[\frac{P(x)+\mathrm{p.v.}I_1(x)}{f_1(x)}\mathrm{sh}\,\tilde{\varphi}(x)+\mathrm{i}\frac{Q(x)+\mathrm{p.v.}I_2(x)}{f_2^+(x)}\mathrm{ch}\,\tilde{\varphi}(x)\right],\quad x\in L_k, \tag{24}$$

$$[u_1'(x)]=-\frac{2\,\mathrm{ch}(\pi\varepsilon)}{p(x)}\left[\mathrm{i}\frac{P(x)+I_1(x)}{f_1^+(x)}\sin\varphi^*(x)+\mathrm{i}\frac{Q(x)+I_2(x)}{f_2^+(x)}\cos\varphi^*(x)\right]+2\mathrm{i}q_2^\infty\delta_1\gamma_* f_t^+(x),\quad x\in M, \tag{25}$$

here the following notations are introduced: $f_1(z)=f(z)$, $f_2(z)=f(z)/Y(z)$; p.v. means principal value of the Couchy's type integrals; $\varphi^*(x)=\varphi^\pm(x)\pm\pi\varepsilon\mathrm{i}$, $x\in M$. Taking boundary values of (18c), one can derive

$$\tilde{\varphi}(x)=-\mathrm{i}Z^+(x)\left[\varepsilon\int_M\frac{\mathrm{d}t}{Z(t)(t-x)}+\mathrm{i}\sum_{k=1}^{(I+J)/2}\int_{L_k}\frac{h_k(t)}{Z^+(t)(t-x)}\mathrm{d}t\right],\quad x\in L, \tag{26a}$$

$$\varphi^*(x)=-Z(x)\left[\varepsilon\int_M\frac{\mathrm{d}t}{Z(t)(t-x)}+\mathrm{i}\sum_{k=1}^{(I+J)/2}\int_{L_k}\frac{h_k(t)}{Z^+(t)(t-x)}\mathrm{d}t\right],\quad x\in M. \tag{26b}$$

Further, for the determination of unknown constants C_k, D_k the coefficients of the following expansions at infinity are required

$$Z(z)=z^{(I+J)/2}[1+\zeta_1/z+\zeta_2/z^2+\mathrm{O}(z^{-3})],\quad Y(z)=z^{(J-I)/2}[1+\eta_1/z+\eta_2/z^2+\mathrm{O}(z^{-3})],$$
$$1/f(z)p(z)=z^{-m}[1+\upsilon_1/z+\upsilon_2/z^2+\mathrm{O}(z^{-3})],\quad \varphi(z)=\alpha_0+\alpha_1/z+\alpha_2/z^2+\mathrm{O}(z^{-3}),$$

in which

$$\eta_1=\frac{1}{2}\left(\sum_{i=1}^{I}a_i-\sum_{j=1}^{J}b_j\right),\quad \varsigma_1=-\frac{1}{2}\left(\sum_{i=1}^{I}a_i+\sum_{j=1}^{J}b_j\right),\quad \upsilon_1=\frac{1}{2}\left(\sum_{k=1}^{N}c_k+\sum_{j=1}^{J}b_j\right)+\sum_{l=2}^{(I+J)/2}d_l,$$

$$\alpha_0=A_{(I+J)/2},\quad \alpha_1=A_{(I+J)/2+1}+A_{(I+J)/2}\zeta_1,\quad \alpha_2=A_{(I+J)/2+2}+A_{(I+J)/2+1}\zeta_1+A_{(I+J)/2}\zeta_2,\ldots$$

$$A_k = \varepsilon \int_M \frac{x^{k-1}\,\mathrm{d}x}{Z(x)} + \mathrm{i}\sum_{j=1}^{(I+J)/2} \int_{L_j} \frac{h_j(x)}{Z^+(x)} x^{k-1}\,\mathrm{d}x = 0, \quad k = (I+J)/2, \ldots$$

Due to the last expansions one has for large $|z|$

$$X(z) = z^{-m}\exp(\mathrm{i}\alpha_0)[1 + \rho_1/z + \rho_2/z^2 + \mathrm{O}(z^{-3})], \quad z \to \infty,$$

where $\rho_1 = \upsilon_1 + \mathrm{i}\alpha_1$, $\rho_2 = \upsilon_2 + \mathrm{i}\alpha_2 + \mathrm{i}\upsilon_1\alpha_1 - \alpha_1^2/2$.

Making use of these expressions, the expansion for $F(z)$ acquires the form

$$\begin{aligned} F(z) = z\exp(\mathrm{i}\alpha_0)\{&C_{m+1} + \mathrm{i}D_{n+1} + [C_m + \mathrm{i}(D_n + D_{n+1}\eta_1) + (C_{m+1} + \mathrm{i}D_{n+1})\rho_1]/z \\ &+ [C_{m-1} + \mathrm{i}(D_{n-1} + D_n\eta_1 + D_{n+1}\eta_2) + (C_m + \mathrm{i}(D_n + D_{n+1}\eta_1))\rho_1 + (C_{m+1} + \mathrm{i}D_{n+1})\rho_2]/z^2\} + \mathrm{O}(z^{-2}). \end{aligned} \tag{27}$$

By substituting this expansion into the condition at infinity (20c) one obtains

$$\begin{cases} C_{m+1} + \mathrm{i}D_{n+1} = \chi_1\exp(-\mathrm{i}\alpha_0), \\ C_m + \rho_1(C_{m+1} + \mathrm{i}D_{n+1}) + \mathrm{i}(D_n + \eta_1 D_{n+1}) = \chi_2\exp(-\mathrm{i}\alpha_0), \end{cases} \tag{28}$$

where $\chi_1 = \mathrm{i}q_2^\infty\delta_1\gamma_0$, $\chi_2 = \tilde{\sigma} - \mathrm{i}\tilde{\tau} + \chi_1\beta_0$.

Considering further the following equality which can be derived from Plemelj's formulae

$$F(z) = \frac{1}{2\pi\mathrm{i}}\int_{L+M} \frac{F^+(x) - F^-(x)}{x - z}\,\mathrm{d}x + \mathrm{const}$$

and expanding the integral at infinity lead to

$$F(z) = \mathrm{const} - \left[\frac{1}{2\pi\mathrm{i}}\int_{L+M}(F^+(x) - F^-(x))\,\mathrm{d}x\right]\frac{1}{z} + \cdots$$

Comparing this expression with the single-valuedness condition (20d) and using (27) give the equation for determination of the constants C_{m-1}, D_{n-1}

$$C_{m-1} + \rho_2(C_{m+1} + \mathrm{i}D_{n+1}) + \rho_1[C_m + \mathrm{i}(D_n + \eta_1 D_{n+1})] + \mathrm{i}(D_{n-1} + D_n\eta_1 + D_{n+1}\eta_2) = \chi_3\exp(-\mathrm{i}\alpha_0), \tag{29}$$

where $\chi_3 = -\mathrm{i}q_2^\infty\delta_1\gamma_*\beta_1$.

The obtained relations (18a)–(18g), (19), (20a)–(20d) represent a complete solution of the combined non-homogeneous Dirichlet–Riemann boundary value problem (17b) which is mathematically correct for any admissible positions of the points b_j. However, in order for the obtained solution describe the solution of the mechanical problem formulated above the following auxiliary conditions should be satisfied

$$[u_2'(b_j)] = 0, \quad \sigma_{22}^{(1)}(x,0) \leqslant 0, \ x \in L, \quad [u_2(x)] \geqslant 0, \ x \in M. \tag{30}$$

The first equation in (30) means that the gap close smoothly as $x \to b_j$. The remaining inequalities indicate that the normal stresses at the contact regions are compressive and there is no interpenetration of the crack surfaces. Using Eq. (23) and the first equation of (30) leads to the following set of transcendental equations for determination of the contact zone lengths defining the real positions of the points b_j

$$P(b_j) + I_1(b_j) = 0. \tag{31}$$

These equations have a number of solutions with respect to each b_j, but only those values of b_j are the solutions of the mechanical problem, which satisfy the inequalities in (30). It should be noted as well that the smooth closure condition is equivalent to the condition of the finiteness of normal stress at the points b_j.

The stress intensity factors (SIFs) at the crack tips a_i can be defined as

$$K_1(a_i) - \mathrm{i}K_2(a_i) = \lim_{x\to a_i}\left[\sigma_{22}^{(1)}(x,0) - \mathrm{i}\sigma_{12}^{(1)}(x,0)\right]\sqrt{|a_i - x|}. \tag{32}$$

Substituting (22) into (33) gives

$$K_1(a_i) = 0, \quad K_2(a_i) = -g(1+\gamma)\frac{Q(a_i) + I_2(a_i)}{f_2^*(a_i)p(a_i)}, \tag{33}$$

where $f_2^*(a_i) = \lim_{x\to a_i}\sqrt{|x - a_i|}f_2(x)$.

5. A single crack with two contact zones

Consider now a particular case of the considered problem when a single crack with two contact zones at the crack tips lies along the interface. In this case one has $I = J = 2$, $N = 0$, $m = n = 2$,

$$h_1(x) = n^*, \quad h_2(x) = \begin{cases} 1, & x \in (b_2; d) \\ 0, & x \in (d; a_2) \end{cases} \ (d = d_2), \quad Y(z) = \sqrt{\frac{(z-b_1)(z-b_2)}{(z-a_1)(z-a_2)}}, \tag{34}$$

$$p(z) = z - d, \quad Z(z) = \sqrt{(z-a_1)(z-b_1)(z-a_2)(z-b_2)}, \quad f(z) = \sqrt{(z-b_1)(z-b_2)}.$$

The function $\varphi(z)$ can be expressed in this case as follows

$$\varphi(z) = \frac{-2}{\sqrt{(a_2-b_1)(b_2-a_1)}}\left[\varepsilon\sqrt{\frac{(z-b_2)(z-a_2)}{(z-b_1)(z-a_1)}}\varphi_1(z) + n^*Y(z)\varphi_2(z) - \frac{\varphi_3(z)}{Y(z)}\right], \tag{35}$$

where

$$\begin{aligned}
&\varphi_1(z) = (b_1-a_1)\Pi(\pi/2, p_1, r') + (z-b_1)K(r'), \quad p_1 = p_1^*\frac{z-a_1}{z-b_1}, \ p_1^* = \frac{b_2-b_1}{b_2-a_1},\\
&\varphi_2(z) = (a_1-a_2)\Pi(\pi/2, p_2, r) + (z-a_1)K(r), \quad p_2 = p_2^*\frac{z-a_2}{z-a_1}, \ p_2^* = \frac{a_1-b_1}{a_2-b_1}, \ r = \sqrt{\frac{\lambda_1\lambda_2}{(1-\lambda_1)(1-\lambda_2)}},\\
&\varphi_3(z) = (b_2-b_1)\Pi(\psi, p_3, r) + (z-b_2)F(\psi, r), \quad p_3 = p_3^*\frac{z-b_1}{z-b_2}, \ p_3^* = \frac{a_2-b_2}{a_2-b_1}, \ r'^2 = 1 - r^2,\\
&\psi = \arcsin\sqrt{\frac{(1-\lambda_1)(d/l - 1/2 + \lambda_2)}{\lambda_2(d/l + 1/2 - \lambda_1)}}.
\end{aligned} \tag{36}$$

Here and in the following $\lambda_1 = (b_1 - a_1)/l$, $\lambda_2 = (a_2 - b_2)/l$, $l = b_1 - a_1$; $F(\psi, r)$, $E(\psi, r)$ and $\Pi(\psi, p, r)$ are the elliptic integrals of the first, second and third kinds, respectively; $K(r)$, $E(r)$, $\Pi(p, r)$ are the complete elliptic integrals.

The expressions for α_0, α_1 and α_2 can be obtained by expending (35) in the series at infinity and can be written in the following form

$$\alpha_0 = \frac{-2}{\sqrt{(1-\lambda_1)(1-\lambda_2)}}\{\varepsilon\lambda_1[\Pi(p_1,r')-K(r')]+n^*[-\Pi(p_2,r)+(1-\lambda_1)K(r)] - (1-\lambda_1-\lambda_2)\Pi(\psi,p_3,r)\},$$

$$\alpha_1 = -\sqrt{(1-\lambda_1)(1-\lambda_2)}\left\{\varepsilon[K(r')-E(r')]+n^*E(r)-E(\psi,r)+\frac{\lambda_2\sqrt{1-r\sin^2\psi}\sin 2\psi}{2(1-\lambda_1-\lambda_2\sin^2\psi)}\right\},$$

$$\alpha_2 = 0.25(\lambda_2-\lambda_1)\sqrt{(1-\lambda_1)(1-\lambda_2)}\left\{\varepsilon[K(r')-E(r')]+n^*E(r)-E(\psi,r) - \frac{\lambda_2\sqrt{1-r\sin^2\psi}\sin 2\psi}{2(1-\lambda_1-\lambda_2\sin^2\psi)^2(\lambda_2-\lambda_1)}\left[1-\lambda_1-\lambda_2\sin^2\psi-2(1-\lambda_1)(1-\lambda_1-\lambda_2)\right]\right\},$$

Eq. (19) for the determination of d in this case takes the following form

$$F(\psi,r)=\omega=\varepsilon K(r')+n^*K(r). \tag{37}$$

Since the elliptic integral $F(\psi,r)$ is positive and $F(\psi,r)<K(r)$ then the following condition for determination of n^* follows from (37)

$$-\varepsilon K(r')/K(r) < n^* < 1-\varepsilon K(r')/K(r).$$

Solving Eq. (37) for ψ and then (36) for d leads to the following expression

$$d=\frac{(0.5-\lambda_2)(1-\lambda_1)+\lambda_2(0.5-\lambda_1)\mathrm{sn}^2(\omega,r)}{1-\lambda_1-\lambda_2\mathrm{sn}^2(\omega,r)},$$

where $\mathrm{sn}(\omega,r)$ is Jacobi elliptic sine-function.

The polynomials $P(z)$ and $Q(z)$ which are determined by the formula (20a) can be presented in the form

$$P(z)=\frac{\tilde\sigma}{\cos\beta}P_1(z)+q_2^\infty\delta_1\gamma_0P_2(z),\quad Q(z)=\frac{\tilde\sigma}{\cos\beta}Q_1(z)+q_2^\infty\delta_1\gamma_0Q_2(z), \tag{38}$$

where

$$P_1(z)=\left(\alpha_1(d-z)-(\alpha_1+d\xi)\frac{\xi}{\xi'}\right)\sin(\alpha_0+\beta)+\left((d-z)\left(\frac{\lambda_1-\lambda_2}{2}-z\right) + \left(d-\alpha_1\xi-\frac{\lambda_1-\lambda_2}{2}\right)\frac{\xi}{\xi'}\right)\cos(\alpha_0+\beta),$$

$$Q_1(z)=\left(z(d-z)+(\alpha_1+d\xi)\frac{\xi}{\xi'}\right)\sin(\alpha_0+\beta)+\left(\alpha_1(d-z)-\left(d-\alpha_1\xi-\frac{\lambda_1-\lambda_2}{2}\right)\frac{1}{\xi'}\right)\cos(\alpha_0+\beta),$$

$$P_2(z)=P^{(1)}(z)\sin(\alpha_0)+P^{(2)}(z)\cos(\alpha_0),\quad Q_2(z)=Q^{(1)}(z)\sin(\alpha_0)+Q^{(2)}(z)\cos(\alpha_0),$$

$$P^{(1)}(z)=\frac{1}{16\xi'}(\xi(8d^2-1-8\xi\alpha_2+\lambda_1(2+\lambda_1)-8d(\alpha_1\xi+\lambda_1-\lambda_2)+2\lambda_2-6\lambda_1\lambda_2+\lambda_2^2 - 4\alpha_1(\alpha_1-\xi(\lambda_1-\lambda_2)))-(d-z)(8z^2-1-4\alpha_1^2+\lambda_1(2+\lambda_1)-8z(\lambda_1-\lambda_2)+2\lambda_2-6\lambda_1\lambda_2+\lambda_2^2) + p(1-\lambda_1-\lambda_2)^2(\xi'(d-z)+\xi)),$$

$$P^{(2)}(z) = \frac{1}{16\xi'}(\xi(8(\alpha_2 + \alpha_1(d - \lambda_1 + \lambda_2)) + \xi(8d^2 - 1 - 4\alpha_1^2 - 4d(\lambda_1 - \lambda_2) + p(1 - \lambda_1 - \lambda_2)^2)) \\ - 8(d - z)(\alpha_2 + \alpha_1(z - \lambda_1 + \lambda_2))\xi'),$$

$$Q^{(1)}(z) = \frac{1}{16\xi'}(4\xi(2\alpha_2 + \alpha_1(2d - \lambda_1 + \lambda_2)) - (8d^2 - 1 - 4\alpha_1^2 - 8d(\lambda_1 - \lambda_2) + \lambda_1(2 + \lambda_1) + \lambda_2(2 + \lambda_2) \\ - 6\lambda_1\lambda_2 + p(1 - \lambda_1 - \lambda_2)^2) + 4(d - z)(2\alpha_2 + \alpha_1(2z - \lambda_1 + \lambda_2))\xi'),$$

$$Q^{(2)}(z) = \frac{1}{16\xi'}(8(\alpha_2 + \alpha_1(d - \lambda_1 + \lambda_2)) + \xi(8d^2 - 1 - 4\alpha_1^2 - 4d(\lambda_1 - \lambda_2)) \\ + (d - z)(4z(2z - \lambda_1 + \lambda_2) - 1 - 4\alpha_1^2)\xi' + p(1 - \lambda_1 - \lambda_2)^2((d - z)\xi' + 8\xi))),$$

where $p = \gamma_*/\gamma_0$, $\tan\beta = \delta = \frac{\tau}{\sigma}$. Expressions for the integrals (18e) and (18f) take the form

$$I_k(z) = q_2^\infty \delta_1 \gamma_0 \tilde{I}_k(z), \quad \tilde{I}_k(x) = -\frac{\cos(\pi n^*)}{\pi}\int_{-a}^{b_1} \frac{I_k^*(t)}{t - x}\,dt - (-1)^k \frac{1}{\pi}\left(\int_{b_2}^{d} \frac{I_k^*(t)}{t - x}\,dt - \int_d^a \frac{I_k^*(t)}{t - x}\,dt\right), \tag{39}$$

where

$$I_1^*(t) = (t - b_1)(t - b_2)(t - d)\,\mathrm{sh}\,\tilde{\varphi}(t), \quad I_2^*(t) = \sqrt{(t - b_1)(t - b_2)(a^2 - t^2)}(t - d)\,\mathrm{ch}\,\tilde{\varphi}(t).$$

Inserting (38_1) and (39_1) into Eq. (31) one can derive a set of two transcendental equations for the determination of the contact zone lengths (or b_j), which can be written in the form

$$\begin{cases} \tilde{q}P_1(b_1) + (1 + \gamma)\gamma_0 \cos\beta[P_2(b_1) + \widetilde{I}_1(b_1)] = 0, \\ \tilde{q}P_1(b_2) + (1 + \gamma)\gamma_0 \cos\beta[P_2(b_2) + \widetilde{I}_1(b_2)] = 0, \end{cases} \tag{40a}$$

where $\tilde{q} = \frac{\sigma}{q_2^\infty \delta_1 g}$. The above equations can be rewritten in the following form, which is much convenient for numerical analysis

$$\frac{P_2(b_1) + \widetilde{I}_1(b_1)}{P_2(b_2) + \widetilde{I}_1(b_2)} = -\frac{P_1(b_1)}{P_1(b_2)}, \quad \frac{\tilde{q}}{(1 + \gamma)\gamma_0 \cos\beta} = -\frac{P_2(b_1) + \widetilde{I}_1(b_1)}{P_1(b_1)}. \tag{40b}$$

For the solution of the system (40b), it is expedient to assign a value of b_1 (b_2) and to find further b_2 (b_1) from the first equation (40b). Then the associated value of $\tilde{q}$ can be found directly from the second formula (40b).

According to Eq. (34) and (38_2) and (39_1), The SIFs can be expressed as

$$\frac{K_2(a_i)}{\sigma\sqrt{l}} = \frac{1}{l(d - a_i)}\left[\frac{Q_1(a_i)}{\cos\beta} + \frac{(1 + \gamma)\gamma_0}{\tilde{q}}[Q_2(a_i) + \widetilde{I}_2(a_i)]\right]. \tag{41}$$

Consider now an easier case when the external shear loading is absent, i.e. $\tau = 0$ and, consequently, $b_2 = -b_1 \equiv b$. The equation for determination of the contact zone lengths (or b) can be obtained in a simpler form if we subtract the first equation of (40a) from the second that leads to

$$b\left[\frac{\tilde{q}}{(1 + \gamma)\gamma_0} - \frac{(\alpha_2 - \alpha_1 d)\cos\alpha_0 + 0.5[b^2(1 + p) - \alpha_1^2]\sin\alpha_0 + J^*(b)}{d\cos\alpha_0 + \alpha_1 \sin\alpha_0}\right] = 0, \tag{42}$$

where

$$J^*(b) = -\frac{\cos(\pi n^*)}{\pi}\int_{-a}^{-b} J_*(x)\,dx + \frac{1}{\pi}\int_b^d J_*(x)\,dx - \frac{1}{\pi}\int_d^a J_*(x)\,dx, \quad J_*(x) = (x - d)\,\mathrm{sh}\,\tilde{\varphi}(x).$$

As can be seen from this equation, there are only three independent dimensionless parameters, namely δ_2/δ_1, ε (or γ) and $\tilde{q}$ which contact zone lengths depend on.

6. One crack with one contact zone

For the case of a crack with one contact zone at the right crack tip one can write similarly to Section 5

$$p(z) = 1, \quad h_k(x) = 0, \quad \alpha_0 = \varepsilon \ln \frac{1 - \sqrt{1-\lambda}}{1 + \sqrt{1-\lambda}}, \quad \alpha_1 = \varepsilon l \sqrt{1-\lambda}, \quad \alpha_2 = -0.25\varepsilon l^2 \lambda \sqrt{1-\lambda},$$

$$\varphi(z) = 2\varepsilon \log \frac{\sqrt{\lambda(z + l/2)}}{\sqrt{z - l/2 + \lambda l} + \sqrt{(1-\lambda)(z - l/2)}}, \quad f(z) = f_t(z) = \sqrt{(z + l/2)(z - l/2 + \lambda l)}.$$

Polynomials $P(z)$ and $Q(z)$ which are determined for a general case by the formula (20a) can be presented by the formulae (38) in which

$$P_1(z) = (z + \lambda l/2)\cos(\alpha_0 + \beta) - \alpha_1 \sin(\alpha_0 + \beta), \quad Q_1(z) = -\alpha_1 \cos(\alpha_0 + \beta) - z \sin(\alpha_0 + \beta),$$

$$P_2(z) = \{2\alpha_1(4z + 3\lambda l)\cos\alpha_0 + [-1 - 4\alpha_1^2 + 8z(z + \lambda l) + 2\lambda l + \lambda^2 l^2 + (1-\lambda)^2 p]\sin\alpha_0\}/8, \tag{43}$$

$$Q_2(z) = \{2\alpha_1(4z + \lambda l)\sin\alpha_0 + [-1 - 4\alpha_1^2 + 8z(z + \lambda l/2) + (1-\lambda)^2 p]\cos\alpha_0\}/8.$$

Expressions for the integrals $I_1(z)$ and $I_2(z)$ in this case take the form

$$\begin{aligned} \widetilde{I}_1(z) &= -\frac{1}{\pi}\int_{0.5l-\lambda l}^{0.5l} \frac{(t + l/2)(t - l/2 + \lambda l)\,\mathrm{sh}\,\tilde{\varphi}(t)}{t - z}\,\mathrm{d}t, \\ \widetilde{I}_2(z) &= \frac{1}{\pi}\int_{0.5l-\lambda l}^{0.5l} \frac{(t + l/2)\sqrt{(t - l/2 + \lambda l)(l/2 - t)}\,\mathrm{ch}\,\tilde{\varphi}(t)}{t - z}\,\mathrm{d}t. \end{aligned} \tag{44}$$

Consistent with the above, the equation for the determination of the relative contact zone length λ acquires the form

$$\frac{\tilde{q}}{(1+\gamma)\gamma_0 \cos\beta} = -\frac{P_2(b) + \widetilde{I}_1(b)}{P_1(b)}, \tag{45}$$

and the SIF can be determined by the formula (41) in which $Q_n(a_i)$ and $\widetilde{I}_2(a_i)$ should be taken from formulae (43) and (44).

7. Results and discussion

Since the influence of the shear loading on the contact zone lengths and the SIFs for a crack with one contact zone (Herrmann and Loboda, 2001) and for the two-contact zone crack without heat flux (Gautesen, 1993) has been studied earlier the numerical analysis is performed here for the case of a single crack with one and two contact zones in tension and heat flux fields only. In Figs. 2 and 3 the behaviour of relative contact zone length $\lambda \equiv \lambda_1 = \lambda_2$ of two-contact zone crack for different thermoelastic parameters ε and $\delta = \delta_2/\delta_1$ and for a wide range of thermomechanical loading parameter $\tilde{q} = \frac{\sigma}{q_2^\infty \delta_1 g l}$ is depicted. As can be seen from these plots when $\tilde{q}$ decreases from infinity to zero (it means that the tension σ is constant and the heat flux q_2^∞ increases from 0 to infinity or q_2^∞ is constant and σ decreases from infinity to 0), the relative contact zone lengths increase (for $\varepsilon > 0$) from small values which correspond to the associated problem

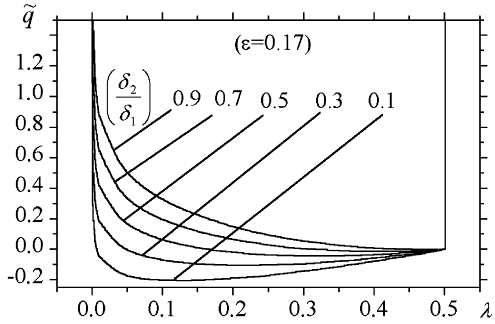

Fig. 2. Dependencies of relative contact zone length λ on thermomechanical loading parameter for different distortivity ratio δ_2/δ_1 and $\varepsilon = 0.17$.

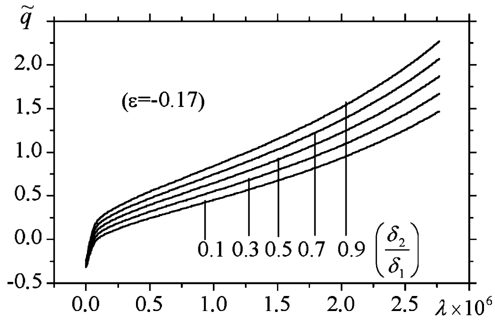

Fig. 3. Dependencies of relative contact zone length λ on thermomechanical loading parameter for different distortivity ratio δ_2/δ_1 and $\varepsilon = -0.17$.

without heat flux to some values which significantly depend on the distortivity ratio δ_2/δ_1. When δ_2/δ_1 approaches 1 then λ approaches 0.5, however, when $\delta_2/\delta_1 = 0$ the relative zone lengths do not depend on $\tilde{q}$ and remain the same as for the problem without heat flux. On the other hand for $\varepsilon < 0$, the values of λ decreases with decreasing $\tilde{q}$ and depends slightly on δ_2/δ_1. As has been shown by Martin-Moran et al. (1983) there are also physically real solutions for negative values of parameter $\tilde{q}$, i.e. for $\sigma < 0$ (q_2^∞ is supposed to be always positive). In the present case the same conclusion holds true, i.e. there are three solutions to Eq. (42) for $\tilde{q} \in [q', 0]$ which satisfy the inequalities (30). One of these solutions is $\lambda = 0.5$ which correspond to the completely closed crack. Here $q'(\varepsilon, \delta_2/\delta_1)$ is a minimum value of $\tilde{q}$ for which Eq. (42) has a single solution which satisfy to Eq. (30). For $\tilde{q} < q'$ and $\varepsilon > 0$, Eq. (42) has no physically real solutions. When $\delta_2/\delta_1 \to 1$ then $q' \to 0$. For $\varepsilon < 0$ the contact zone lengths decrease with decreasing $\tilde{q}$.

The dependencies of the dimensionless SIF $K^* = K_2/\sigma\sqrt{l}$ at the right crack tip on $\tilde{q}$ for different parameters of distortivities are depicted in Fig. 4. For positive values of $\tilde{q}$ the absolute value of K^* increases up to infinity in decreasing $\tilde{q}$ to zero for both positive and negative values of ε, and it do not depend significantly on thermoelastic properties of bimaterial.

In many papers an interface crack with a single contact zone at the crack tip is considered for the sake of simplicity. According to Gautesen and Dundurs (1987) the ignoring of another contact zone leads to a negligible small error in the longer contact zone length determination and the associated SIF for a pure mechanical loading. The results of the correspondent analysis for a thermomechanical loading are presented in Tables 1 and 2, where the values of $\tilde{q}$ are given for a crack with one ($\tilde{q}_{\mathrm{I}}$) and two ($\tilde{q}_{\mathrm{II}}$) contact zones.

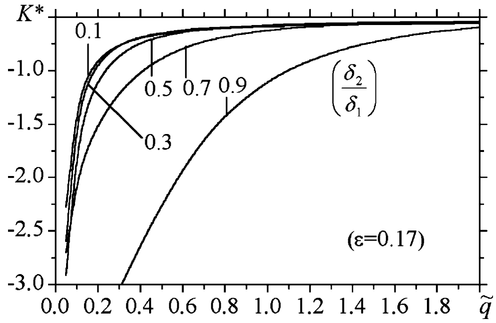

Fig. 4. Dependencies of SIF on thermomechanical loading parameter for different distortivity ratio δ_2/δ_1 and $\varepsilon = 0.17$.

Table 1
Thermomechanical loading parameter values $\tilde{q}_{\mathrm{I}}$ for $\varepsilon = 0.17$

δ	λ			
	0.001	0.01	0.05	0.1
0.1	0.334326	−0.0437699	−0.171459	−0.205251
0.3	0.691907	0.190177	0.00968267	−0.048829
0.5	1.04949	0.424124	0.190825	0.107592
0.7	1.40707	0.658072	0.371967	0.264014
0.9	1.76465	0.892019	0.553109	0.420435

Table 2
Thermomechanical loading parameter values $\tilde{q}_{\mathrm{II}}$ for $\varepsilon = 0.17$

δ	λ			
	0.001	0.01	0.05	0.1
0.1	0.333324	−0.0472149	−0.175294	−0.202242
0.3	0.690335	0.183112	0.08803153	−0.0701398
0.5	1.04735	0.413438	0.159231	0.061962
0.7	1.40436	0.643765	0.326493	0.194064
0.9	1.76137	0.874091	0.493755	0.326166

The correspondent dimensionless SIFs K_{I}^* (evaluated for a crack with one contact zone) and K_{II}^* (for a crack with two contact zones) evaluated for the values of the thermomechanical loading parameter $\tilde{q}_{\mathrm{II}}$ given in Table 1 are presented in Tables 3 and 4.

Table 3
SIFs K_{I}^* evaluated for values of $\tilde{q}_{\mathrm{II}}$ from Table 2 and λ from Table 5 for $\varepsilon = 0.17$

δ				
0.1	−1.212	5.040	1.019	0.7719
0.3	−0.7713	−1.719	29.21	2.687
0.5	−0.6312	−0.9476	−1.834	−3.706
0.7	−0.5624	−0.7282	−1.072	−1.410
0.9	−0.5214	−0.6244	−0.8270	−0.9772

It can be clearly seen from these tables that for small values of relative contact zones ($\lambda < 0.01$) the values of SIFs and relative contact zones evaluated at the assumption that a crack has one and two contact zones are practically the same. However for essential contact zone lengths the difference of the results

Table 4
SIFs K_{II}^* evaluated for values of $\tilde{q}_{II}$ from Table 2 for $\varepsilon = 0.17$

δ	λ			
	0.001	0.01	0.05	0.1
0.1	−1.211	4.972	0.9283	0.5959
0.3	−0.7710	−1.701	27.33	2.348
0.5	−0.6310	−0.9389	−1.735	−3.371
0.7	−0.5622	−0.7222	−1.020	−1.304
0.9	−0.5213	−0.6197	−0.7892	−0.9112

Note that in evaluating K_I^*, the correspondent values of λ, evaluated from Eq. (45) for a crack with one contact zone and presented in Table 5, were used.

Table 5
Relative contact zone lengths λ (for a crack with one contact zone) evaluated for the values of $\tilde{q}_{II}$ from Table 2 and $\varepsilon = 0.17$

0.1	0.001004	0.01035	0.05339	0.1142
0.3	0.001004	0.01054	0.06083	0.1351
0.5	0.001005	0.01065	0.06456	0.1511
0.7	0.001005	0.01072	0.06675	0.1583
0.9	0.001005	0.01077	0.06817	0.1623

presented in Tables 1 and 2 is rather tangible, especially for δ tending to 1. The correspondent differences of the values K_I^* and K_{II}^* given in Tables 3 and 4 are not so essential, however for $\lambda = 0.1$ it is rather tangible.

8. Conclusion

A plane problem for a set of interface cracks in an infinite isotropic bimaterial under the action of remote mixed mode mechanical loading and a heat flux is considered. An exact analytical solution, which takes into account contact zones at both crack tips, is found. The set of transcendental equations (37) for the determination of the contact zone lengths is formulated and the analytical formulae for the associated stress intensity factors are given. It is shown that the relative contact zone lengths and the SIFs for an interface crack under a combined tension–shear (σ–τ) field and a uniform heat flux (q_2^∞) depend on two dimensionless thermomechanical parameters τ/σ, $\tilde{q} = \frac{\sigma}{q_2^\infty \delta_1 g l}$ and two thermoelastic parameters ε, δ_2/δ_1.

A numerical analysis has been performed for a single crack with one and two contact zones under the action of a tensile loading and a heat flux. It follows from the obtained results that for $0 < \delta_2/\delta_1 \leqslant 1$ and $\varepsilon > 0$ the contact zone lengths increase in decreasing $\tilde{q}$ from $+\infty$ to some negative limiting value, which depends on the loading and the thermoelastic parameters of bimaterial. As soon as $\tilde{q}$ becomes smaller than the mentioned limiting value the crack close abruptly. For negative values of $\tilde{q}$ larger then the negative limiting values, three possible contact zone lengths and the associated SIFs exist which depend on the history of the loading. For $\varepsilon < 0$, contact zones decrease in decreasing $\tilde{q}$ from $+\infty$ to some negative values for which crack faces get into contact at the central (if the shear loading is absent) part of the crack, so that a new contact zone is created.

The comparison of the results obtained in the framework of the models of an interface crack with one (Tables 1, 3 and 4) and two (Tables 2 and 5) contact zones has been performed. It appears that for a rather long contact zones ($\lambda > 0.01$) the difference in the associated results is sensitive, but for the most practically common cases ($\lambda \leqslant 0.01$) the mentioned difference is negligibly small and the model of a crack with one contact zone can be used for the determination of the contact zone length and the associated stress intensity factor.

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